Student seminar solutions Week 11

1. We first want to find the valuation $v_p(n!)$ for any number $n \in \mathbb{N}$. First, notice that if $n < p$, then $v_p(n!) = 0$. Now if $n \geq p$, then we have contributions of powers of p in $n!$:

$$p, 2p, \dots, \lfloor \frac{n}{p} \rfloor p$$

but also (if $n \geq p^2$) contributions given by

$$p^2, 2p^2, \dots, \lfloor \frac{n}{p^2} \rfloor p^2$$

and so on with powers of p.
Hence, the numbers of powers of p in the decomposition in prime number of $n!$ is given by

$$v_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$$

where this sum is finite since after some large $k \in \mathbb{N}$, $\lfloor \frac{n}{p^k} \rfloor = 0$. Notice that since $\lfloor \frac{n}{p^k} \rfloor \leq \frac{n}{p^k}$ for any $k \in \mathbb{N}$,

$$v_p(n!) \leq \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1}$$

since it's a geometric serie. Hence,

$$|\frac{1}{n!}|_p = p^{-v_p(\frac{1}{n!})} = p^{v_p(n!)} \leq p^{\frac{n}{p-1}}$$

By real analysis, the radius of convergence $R \in \mathbb{Q}$ satisfies

$$\frac{1}{R} = \limsup_{n \to \infty} |\frac{1}{n!}|^{\frac{1}{n}} \leq \limsup_{n \to \infty} (p^{\frac{n}{p-1}})^{\frac{1}{n}} = p^{\frac{1}{p-1}}$$

So $R \geq p^{-\frac{1}{p-1}}$.

Now consider $x = p^{\frac{1}{p-1}}$ as a p-adic number in $\mathbb{C}_p$, so that $v_p(x) = \frac{1}{p-1}$. We want to show that $\exp(x)$ doesn't converge. First, for $i \in \mathbb{N}$, notice that

$$v_p(p^i!) = \sum_{k=1}^{\infty} \lfloor \frac{p^i}{p^k} \rfloor = \sum_{k=1}^{i} \frac{p^i}{p^k} = p^i \sum_{k=1}^{i} p^{-k} = p^i \frac{1 - p^{-i}}{p-1} = \frac{p^i - 1}{p-1}$$

Hence,

$$v_p(\frac{x^{p^i}}{p^i!}) = p^i v_p(x) - v_p(p^i!) = \frac{p^i}{p-1} - \frac{p^i-1}{p-1} = \frac{1}{p-1}$$

Then, $|\frac{x^{p^i}}{p^i!}|_p = p^{\frac{1}{p-1}}$ is a constant for any $i \in \mathbb{N}$, which shows that $\lim_{n\to\infty} \frac{x^n}{n!} \neq 0$, so the serie cannot converge. Finally since $|x|_p = p^{-\frac{1}{p-1}}$, $R = p^{-\frac{1}{p-1}}$.

2. We can view k_2 as a k_1-vector space, and we choose $\{e_1, \dots, e_n\}$ to be a basis, where $n = [k_2 : k_1]$. Recall that the norm map $N_{k_2/k_1} : k_2 \to k_1$ is defined as $N_{k_2/k_1}(x) = \det(M_x)$ the determinant of the map $M_x : k_2 \to k_2$ that sends $y \mapsto xy$, seen as linear map. In particular, if for $x \in k_2$ we write $x = x_1e_1 + \cdots + x_ne_n$ with $x_i \in k_1$, notice that $N_{k_2/k_1}(x) = P(x_1, \dots, x_n)$ where P is a polynomial in $k_1[X_1, \dots, X_n]$. Hence, $N_{k_2/k_1}(x)$ is the composition of a projection function, addition and multiplication in k_1. We just need to show that addition and multiplication are continuous with respect to the p-adic topology.

   Without loss of generality, we show continuity aroud 0. The set $B(0,\epsilon) = \{x \in k_1 \mid |x|_p < \epsilon\}$ give a basis of the open set around 0 in k_1. Let $+ : k_1 \times k_1 \to k_1$ be the addition map and $\cdot : k_1 \times k_1 \to k_1$ be the multiplication.

   Take $(x, y) \in B(0,\epsilon) \times B(0,\epsilon)$. Since $|\cdot|_p$ is a non archimedean norm,

$$|x + y|_p = \max\{|x|_p, |y|_p\} < \epsilon$$

   which shows that $B(0,\epsilon) \times B(0,\epsilon)$ is contained in the preimage of $B(0,\epsilon)$ by $+$, hence $+$ is continuous.

   Take $(x, y) \in B(0,\sqrt{\epsilon}) \times B(0,\sqrt{\epsilon})$. Then

$$|xy|_p = |x|_p|y|_p \leq \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon$$

   so $B(0,\sqrt{\epsilon}) \times B(0,\sqrt{\epsilon})$ is contained in the preimage of $B(0,\epsilon)$ by $\cdot$, hence $\cdot$ is continuous.

3. (i) Recall that we have

$$\mathcal{I}_K(\mathfrak{m}) = \{\mathfrak{a} \in \mathcal{I}_K : \text{ord}_\mathfrak{p}\mathfrak{a} = 0 \text{ for all } \mathfrak{p}|\mathfrak{m}\}$$

$$\mathcal{J}^+_{K,\mathfrak{m}} = \{\mathbf{a} \in \mathcal{J}_K : a_v > 0 \text{ for all real } v \text{ and } a_v \equiv 1 \pmod{\mathfrak{p}_v^{\text{ord}_v(\mathfrak{m})}} \text{ for all } \mathfrak{p}_v|\mathfrak{m}\}$$

   Take $\mathfrak{a} \in \mathcal{I}_K(\mathfrak{m})$ and let

$$\mathfrak{a} = \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v\mathfrak{a}} = \prod_{v \text{ finite},\ \mathfrak{p}_v \nmid \mathfrak{m}} \mathfrak{p}_v^{\text{ord}_v\mathfrak{a}}$$

be its decomposition in primes, since $\text{ord}_{\mathfrak{p}}\mathfrak{a} = 0$ for all $\mathfrak{p}|\mathfrak{m}$. For every $v \in V_K$ such that $\mathfrak{p}_v \nmid \mathfrak{m}$, choose $a_v \in K_v^\times$ such that $\text{ord}_v(a_v) = \text{ord}_v(\mathfrak{a})$ and take $a_v = 1$ for the $v \in V_K$ such that $\mathfrak{p}_v|\mathfrak{m}$. By taking

$$\mathbf{a} = (\dots, a_v, \dots)_{v \in V_K}$$

$\eta_{K,\mathfrak{m}}(\mathbf{a}) = \mathfrak{a}$.

(ii)-iii) Let $x, y \in K^\times$ two units such that $x\mathbf{a}, y\mathbf{a} \in \mathcal{J}_{K,\mathfrak{m}}^+$. Consider $\frac{x}{y} \in K^\times$. Notice the following:

(a) For any real place $v \in V_K,\ ya_v > 0,\ xa_v > 0$ by definition of $\mathcal{J}_{K,\mathfrak{m}}^+$, so $\frac{x}{y} = \frac{xa_v}{ya_v} > 0$

(b) For any place $v \in V_K$ such that $\mathfrak{p}_v|\mathfrak{m}$,

$$1 \equiv xa_v \equiv \frac{x}{y} ya_v \equiv \frac{x}{y} \pmod{\mathfrak{p}_v^{\text{ord}_v \mathfrak{m}}}$$

by definition of $\mathcal{J}_{K,\mathfrak{m}}^+$.

Together, we have that $\frac{x}{y} \in \mathcal{J}_{K,\mathfrak{m}}^+$, and so $(\frac{x}{y}) \in \mathcal{P}_{K,\mathfrak{m}}^+$. Now

$$\alpha(\mathbf{a}) = \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(xa_v)} = \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(\frac{x}{y} ya_v)} = \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(\frac{x}{y}) + \text{ord}_v(ya_v)}$$

$$= \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(ya_v)} \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(\frac{x}{y})} = \prod_{v \text{ finite}} \mathfrak{p}_v^{\text{ord}_v(ya_v)} (\frac{x}{y})$$

Hence it is well defined in the quotient.

(iv) From proposition 4.3.3 in Childress' book, the kernel of α is

$$\ker(\alpha) = K^\times \mathcal{E}_{K,\mathfrak{m}}^+$$

4. This exercice is a generalization of proposition 4.5.6 of Childress' book for arbitrary (non necessarily Galois) finite extension of field. The only part of this proof that uses the Galois assumption is that for a finite place $v \in V_F$, and if $w \mid v$ is ramified, then $[\,\mathcal{U}_v :\ N_{K_w/F_v}(\mathcal{U}_w)\,] \leq e(v/w)$ (from corollary 5.5). We want to prove that this index is always finite even for non Galois extension, and the rest of the proof is the same.

   Let $\mathcal{U}_v^{(n)} = 1 + \mathfrak{p}_v^n \subseteq \mathcal{U}_v$ be the n-th higher unit group. Notice that we have a tower of groups

$$\cdots \subset \mathcal{U}_v^{(n+1)} \subset \mathcal{U}_v^{(n)} \subset \cdots \subset \mathcal{U}_v^{(1)} \subset \mathcal{U}_v^{(0)} = \mathcal{U}_v$$

Since K_w/F_v is a finite extension, there is a number $n \in \mathbb{N}$ such that

$$[\,\mathcal{U}_v^{(n)} : N_{K_w/F_v}(\mathcal{U}_w)\,] < +\infty$$

Moreover, from (**?**, Proposition 3.10), we have that

$$\mathcal{U}_v^{(i)} / \,\mathcal{U}_v^{(i+1)} \cong \mathcal{O}_v/\mathfrak{p}_v$$

which is a finite field from lecture. Hence $[\,\mathcal{U}_v^{(i)} : \,\mathcal{U}_v^{(i+1)}\,] < +\infty$ for every $i \in \mathbb{N}$. Finally we have that

$$[\,\mathcal{U}_v : \,N_{K_w/F_v}(\mathcal{U}_w)\,] = [\,\mathcal{U}_v : \mathcal{U}_v^{(1)}\,][\,\mathcal{U}_v^{(1)} : \mathcal{U}_v^{(2)}\,]\cdots[\,\mathcal{U}_v^{(n-1)} : \mathcal{U}_v^{(n)}\,][\,\mathcal{U}_v^{(n)} : \,N_{K_w/F_v}(\mathcal{U}_w)\,] < +\infty$$